

# Thermal resistance of bridged cracks in fiber-reinforced ceramic composites

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The thermal resistance of a bridged matrix crack in a fiber-reinforced ceramic composite is analyzed. The problem is cast in terms of a unit cell comprising an infinitely long composite cylinder with a single matrix crack perpendicular to the fiber axis. At the outset, it is demonstrated that the thermal resistance of such a crack can be represented by a simple circuit consisting of two parallel resistors; one resistor represents the thermal resistance of the gas phase  $R_g$  within the matrix crack, and the other resistor represents the constriction resistance  $R_c$  of the bridging fiber. The main focus of the article is on determination of  $R_c$  and bounds on this resistance are obtained by the use of variational calculus. The analogy between problems involving steady-state heat flow and elasticity in multiphase materials is emphasized. The results for the constriction resistance are compared with the predictions of an approximate analytical model presented by [T. J. Lu and J. W. Hutchinson, *Philos. Trans. R. Soc. London, Ser. A* **351**, 595 (1995)]. In their model the radial temperature variation within the matrix is neglected. The domain in which such variations can be justifiably neglected is found. © 2001 American Institute of Physics. [DOI: 10.1063/1.1354648]

## I. INTRODUCTION

One of the main mechanisms by which fiber-reinforced ceramic composites obtain high toughness is through the development of multiple matrix cracks accompanied by the associated debonding and sliding that occur along the fiber-matrix interface adjacent to each crack.<sup>1</sup> The resulting inelastic strain imparts a high toughness to the composite in essentially the same manner that dislocation plasticity imparts high toughness to metallic alloys. However, these cracks are detrimental to other properties of interest in engineering design. For instance, they lead to permanent residual strain which compromises dimensional stability. Moreover, they impede the flow of heat perpendicular to the crack plane and therefore reduce the average composite conductivity.<sup>2-4</sup> The latter problem is the focus of the present article.

The specific problem of interest involves an aligned fiber composite that contains a periodic array of fully bridged matrix cracks perpendicular to the fiber axis as shown schematically in Fig. 1. The reduction in the longitudinal thermal conductivity due to the cracks is dictated by the spacing of the cracks and the thermal resistance of each crack. In turn, this thermal resistance is controlled by the pattern of heat flow along with the pertinent thermal and geometric properties of the constituent phases. The reduced thermal conductivity  $K_z$  can be expressed in terms of: (i) the thermal resistance, or its inverse the conductance  $H_c$ , of a single isolated crack; (ii) the crack spacing  $d$ ; and (iii) the axial thermal conductivity  $k_z$  of the pristine composite. A series resistance summation yields the relation<sup>4</sup>

$$\frac{1}{K_z} = \frac{1}{k_z} + \frac{1}{dH_c}. \quad (1)$$

The challenge then is to determine the thermal conductance of a single crack in terms of the pertinent fiber, matrix, and interface properties. This goal can be accomplished most conveniently through an analysis of a unit cell with periodic boundary conditions that simulate the conditions obtained in the actual composite. This approach is adopted here.

This exact problem was analyzed recently by Lu and Hutchinson<sup>5</sup> following an approach reminiscent of shear lag in the context of mechanical loading. For the sake of convenience, reference to their work is denoted by the abbreviation "L.H." In order to obtain closed-form analytical solutions to the governing heat flow equations, L.H. neglected the radial temperature variations within the matrix near the crack plane. (Analogously, the radial variation in the normal matrix strain is neglected in the standard shear lag formulation of the longitudinal elastic modulus of the cracked composite.) In the domain in which these temperature variations can be neglected, their model provides useful results for the relationship between the composite conductivity and the pertinent constituent properties.

The L.H. treatment, as well as the analysis given here, is based upon the thermal field in a unit cell. The cell, which is discussed in more detail in the next section, has a diameter of  $2b$ , which is equal to the distance between fiber centers. The cell length  $d$  is equal to the crack spacing. The cell is bounded at each end by cracks; the heat flux enters and exits the cell across these cracks. The disturbance to the flow, caused by the crack-induced constriction, decays approximately within a distance equal to the diameter. In most composites of practical interest, the crack spacing  $d \gg 2b$  so that there is virtually no thermal interaction between cracks. For

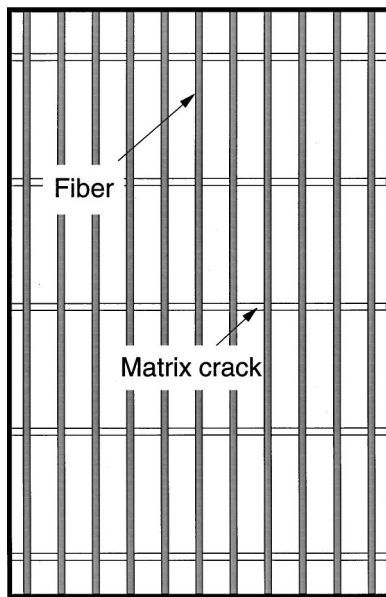


FIG. 1. Aligned fiber composite showing the fiber orientation and the matrix cracks. The overall heat flow is along the fiber direction.

this reason, the interactions are neglected in the present article and attention is focused upon a single crack.

The main objective of the present article is to develop solutions for the thermal resistance of an isolated bridged crack in an aligned fiber composite. The solutions presented here are more rigorous than those of L.H. in the sense that they account for the radial temperature variations in the matrix, and are therefore applicable to a broader range of constituent properties. In the first step of the analysis, it is demonstrated that the combined effect of the thermal resistance of the gas phase within the matrix crack plus the constriction resistance of the bridging fiber, can be obtained from a simple parallel resistor model. The essence of the problem, and the focus of the majority of this article, is the determination of the constriction resistance. For this purpose, an approach based on variational calculus is used to obtain upper and lower bounds on the constriction resistance. Emphasis is placed on the analogy between problems involving steady-state heat flow and elasticity of multiphase materials.

It is noted parenthetically that, in the present article, the key results are cast in terms of the constriction resistance. The constriction resistance can be combined with the thermal resistance of the gas phase through the aforementioned parallel resistor model to obtain the average crack conductance  $H_c$ . This crack conductance  $H_c$ , in turn, can be combined with the crack density to obtain the composite conductivity  $K_z$ , via Eq. (1), and this conductivity is the property of interest at the macroscopic scale. In this context, the constriction resistance is arguably the key fundamental property associated with fiber bridging. By contrast, L.H. couch their results in terms of composite conductivity. In the latter form, the results provide a somewhat less direct measure of the effect of fiber bridging. Despite this difference in viewpoints, the connections between Lu and Hutchinson's results and the ones presented here can be readily made, as illustrated later.

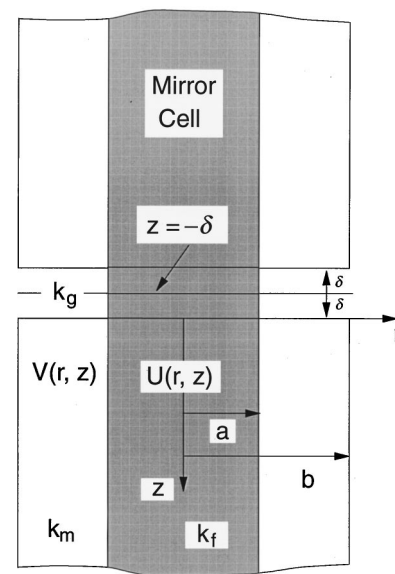


FIG. 2. Unit cell, with radius  $b$  containing a fiber of radius  $a$ , occupying the region  $0 < z < \infty$ . The matrix crack has total thickness equal to  $2\delta$  and is bridged by the fiber. The mirror cell is shown, and, by virtue of symmetry, the surface  $z = -\delta$  is isothermal. The gas conductivity is  $k_g$ .

## II. UNIT CELL AND BOUNDARY CONDITIONS

The effects of a matrix crack upon the axial heat flow in a unidirectional fiber composite are investigated using the unit cell shown in Fig. 2. The cell to be analyzed is the circular cylinder of radius  $b$ , occupying the region  $0 < z < \infty$ . The cell contains a fiber of radius  $a$  so that the volume fraction of the fiber is  $f = a^2/b^2$ . The matrix crack, which has a total thickness  $2\delta$ , is centered at  $z = -\delta$ , and the gas conductivity within the crack is equal to  $k_g$ . The crack thickness is small in comparison with the cell diameter, i.e.,  $b \gg \delta$ , and, as shown, the fiber bridges the crack. There is a "mirror" cell above the crack, and, by virtue of symmetry, the plane  $z = -\delta$  is an isothermal surface.

The temperatures in the fiber and matrix are designated by  $U(r, z)$ ,  $V(r, z)$ , respectively, and the thermal conductivities of these two phases are designated  $k_f$ ,  $k_m$ . The heat flow is axisymmetric so that both  $U$  and  $V$  must satisfy Laplace's heat flow equation

$$\nabla^2 U = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{\partial^2 U}{\partial z^2} = 0, \quad a > r > 0, \quad (2a)$$

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial z^2} = 0, \quad b > r > a. \quad (2b)$$

It is assumed that the flux is related to the temperature gradient by Fourier's Law.

Although this circular cell does not fill space, it is reasonably close to a hexagonal prism. Since the cell is part of a periodic array, there is no radial flux at  $r = b$ , and therefore it follows that

$$\frac{\partial V}{\partial r} = 0. \quad (3)$$

The cell can be regarded as a "heat pipe" in the sense that there is no leakage at  $r = b$ .

At  $r=a$  the radial flux must be continuous. In contrast, there is a temperature discontinuity across the interface. The boundary conditions are

$$k_f \frac{\partial U}{\partial r} = k_m \frac{\partial V}{\partial r}, \tag{4a}$$

$$k_f \frac{\partial U}{\partial r} = h(V-U), \tag{4b}$$

where  $h$  is the conductance of the interface between the two phases. Most of the resistance to the heat exchange between the two phases is due to the interface when  $h$  becomes sufficiently small. Moreover, the ratio, of the fiber resistance to that of the interface, is a dimensionless quantity that is equal to the Biot number

$$Bi_f = \frac{ah}{k_f}. \tag{5}$$

Similarly, within the the matrix, the Biot number can be defined as

$$Bi_m = \frac{(b-a)h}{k_m}. \tag{6}$$

When both of these Biot numbers are sufficiently small, the temperature variation in the radial direction is negligible, and the analysis then undergoes a significant simplification. Generally, however, the Biot numbers are not small enough to make this assumption and a more rigorous solution is necessary.

The total amount of heat flowing down the heat pipe is equal to  $Q$  so that  $P = Q/\pi b^2$  is the average flux. At distances from the crack that are large in comparison with the pipe diameter, the heat flow becomes axial. The flux  $P$  is split;  $P_f$  is the flux in the fiber, and  $P_m$  is the flux in the matrix. These fluxes are

$$P_f = \frac{k_f P}{k_z}, \tag{7}$$

$$P_m = \frac{k_m P}{k_z},$$

where the average axial conductivity  $k_z$  is given by the law of mixtures

$$k_z = k_f f + k_m (1-f). \tag{8}$$

The matrix crack impedes the axial flow of heat and causes a certain degree of constriction. The constriction is not total, however, because the conductance  $h_c$  across the crack is generally finite. Near the crack a portion of the heat flow is shunted into the fiber, and to account for this partial constriction a parameter  $C$  is introduced. The flux boundary condition, imposed upon the cell at  $z=0$ , is written in the form

$$\sigma_f = k_f \frac{\partial U}{\partial z} \Big|_0 = P_f + CS_f, \tag{9}$$

$$\sigma_m = k_m \frac{\partial V}{\partial z} \Big|_0 = P_m + CS_m.$$

The quantities  $S_f$  and  $S_m$ , which contribute no net axial flow, are

$$S_f = \frac{P_m(1-f)}{f},$$

$$S_m = -P_m. \tag{10}$$

The parameter  $C$ , which varies between  $0 \leq C \leq 1$ , describes the relative degree of constriction. When  $C=1$ , then  $\sigma_m=0$ ,  $\sigma_f=P/f$ , and this corresponds to complete constriction. At the other extreme, when  $C=0$ , then  $\sigma_f=P_f$ ,  $\sigma_m=P_m$ , and this corresponds to no constriction. Since the crack conductance invariably lies in the range  $\infty > h_c > 0$ , the parameter  $C$  assumes some intermediate value.

Evidently, the flux boundary condition in (9) is split into two components. The first term in (9) represents uniform axial flow, and the second term, containing the parameter  $C$ , describes the constriction. Since the problem is linear, it is possible to use the method of superposition and write the temperatures as the sum of these two contributions

$$U(r,z) = \frac{Pz}{k_z} - C\mathcal{U}(r,z), \tag{11}$$

$$V(r,z) = \frac{Pz}{k_z} - C\mathcal{V}(r,z).$$

The first terms on the right-hand side represent uniform axial flow while the unknown functions,  $\mathcal{U}(r,z)$  and  $\mathcal{V}(r,z)$ , describe the constriction effects. These functions must satisfy Laplace's Eqs. (2a) and (2b) along with the boundary conditions (3) and (4a) and (4b). The flux condition at  $z=0$  is

$$k_f \frac{\partial \mathcal{U}}{\partial z} \Big|_0 = -S_f, \tag{12}$$

$$k_f \frac{\partial \mathcal{V}}{\partial z} \Big|_0 = -S_m.$$

It is easily shown that  $\pi a^2 S_f + \pi (b^2 - a^2) S_m = 0$ . Using this, along with adiabatic condition at  $r=b$ , it follows from heat flow conservation that

$$\frac{\partial}{\partial z} \left[ 2\pi k_f \int_0^a \mathcal{U}(r,z) r dr + 2\pi k_m \int_a^b \mathcal{V}(r,z) r dr \right] = 0. \tag{13}$$

The portion within the brackets  $[\dots]$  must be constant. At distances  $z \gg b$ , both  $\mathcal{U}$  and  $\mathcal{V}$  decay to a constant temperature which can be taken to be zero. Thus, it follows that the portion within the brackets  $[\dots]$  is equal to zero at any position along the length.

### III. DETERMINING THE PARAMETER $C$

The constriction parameter  $C$  can be found using the formal analogy that exists between steady state heat flow and linear elasticity.<sup>6</sup> In this analogy, temperature, flux, and thermal conductivity play the role of displacement, stress, and elastic stiffness, respectively.

**A. “Energy” associated with constriction**

The “strain energy”  $E_s$  associated with the heat flow is equal to half the volume integral of the flux multiplied by the temperature gradient. Thus, over a length  $0 < z < L$ , the strain energy in the heat pipe is

$$E_s = \pi \int_0^L \left[ k_f \int_0^a (U_r^2 + U_z^2) r dr + k_m \int_a^b (V_r^2 + V_z^2) r dr \right] dz + \pi h a \int_0^L [U(a, z) - V(a, z)]^2 dz, \tag{14}$$

where the subscripts  $r$  and  $z$  represent partial differentiation. The partial derivatives are the “strains,” and the thermal conductivities are the “stiffness coefficients.” The interface conductance  $h$  is a “spring constant” and  $[V - U]$  is the spring “displacement.”

Upon substituting the expressions in Eqs. (11) into (14), and integrating, it follows that  $E_s$  assumes the form

$$E_s = E_0 + E_1 C + E_2 C^2.$$

The first term  $E_0$  is due to the uniform flow and it is easily verified that

$$E_0 = \frac{Q^2 R_0}{2},$$

where  $R_0 = L / \pi b^2 k_z$ . This is the thermal resistance of uncracked material of length  $L$ ; in the analogy, the thermal resistance is equivalent to the compliance. The second term  $E_1 C$  represents the thermal interaction between the uniform flow and the constriction. Upon substituting (11) into (14) it follows that

$$E_1 = -\frac{P}{k_z} \int_0^L \left( 2\pi k_f \int_0^a \frac{\partial U}{\partial z} r dr + 2\pi k_m \int_a^b \frac{\partial U}{\partial z} r dr \right) dz = 0.$$

The result that  $E_1 = 0$  follows from Eq. (13). Finally, the term  $C^2 E_2$  represents the energy due to the constriction effects. The form of  $E_2$  is identical to  $E_s$  except that the temperatures  $U, V$  are replaced by  $\mathcal{U}, \mathcal{V}$  respectively. As  $L$  becomes large the energy  $E_2$  approaches a limiting value. Using the properties placed upon  $\mathcal{U}$  and  $\mathcal{V}$  it follows, after integrating by parts, that as  $L \rightarrow \infty$ :

$$E_2 = \frac{Q \bar{\mathcal{U}}}{2},$$

where  $\bar{\mathcal{U}}$  is the average temperature at  $z = 0$ . The constriction resistance is defined as  $R_c = \bar{\mathcal{U}} / Q$ , so that, when  $L \gg b$  it is possible to write

$$E_s = \frac{Q^2}{2} (R_0 + C^2 R_c). \tag{15}$$

To further illustrate the elasticity/thermal analogy consider the behavior corresponding to the limiting values for  $C$ . When  $C = 0$  the heat flow is uniform and the strain energy  $E_s = Q^2 R_0 / 2$  corresponds to the case of a uniformly stressed bar of length  $L$ . The other limit, when  $C = 1$ , corresponds to a cracked specimen and the strain energy is  $E_s = Q^2 (R_0 + R_c) / 2$ . The constriction resistance is the analog of the increase in the compliance caused by the crack. Unlike  $R_0$

which is unbounded as  $L \rightarrow \infty$ , the constriction resistance  $R_c$  approaches a finite limit. Henceforth, in this article, the constriction resistance  $R_c$  is understood to be this limiting value as  $L \rightarrow \infty$ . The immediate goal in this section is to find how the axial flux distributes itself across the plane of the crack. For this purpose, it is necessary to find the energy associated with heat flow across the air gap.

**B. Energy associated with heat flow through the gas phase**

The energy associated with heat flow through the gas is considered now. By virtue of symmetry the surface  $z = -\delta$  is isothermal and, provided  $b \gg \delta$ , the heat flow through the region  $-\delta < z < 0$  is predominantly one dimensional. The volume of the gap is  $\pi b^2 \delta$  and the strain energy in the gap  $E_g$  is

$$E_g = \delta \pi b^2 \left[ \frac{f \sigma_f^2}{2k_f} + \frac{(1-f) \sigma_m^2}{2k_g} \right].$$

The crack conductance and gas conductivity are related by  $k_g = 2h_c \delta$ , and since  $k_g \ll k_f$  the energy can be approximated by  $E_g \approx \pi b^2 (1-f) \sigma_m^2 / 4h_c$ . Using the expression for  $\sigma_m$  given in Eq. (9) it follows that  $E_g$  is

$$E_g = \frac{Q^2}{2} (1-C)^2 R_g, \tag{16}$$

where the effective thermal resistance of the gas  $R_g$  is

$$R_g = \frac{(1-f) k_m^2}{2 \pi b^2 k_z^2 h_c}. \tag{17}$$

The energy  $E_g$  vanishes when  $C = 1$  and this corresponds to no heat flow across the gas.

**C. Total resistance due to constriction and gas**

The quantity  $E = E_s + E_g$  gives the total energy of half the cell

$$E = \frac{Q^2}{2} [R_0 + C^2 R_c + (1-C)^2 R_g]. \tag{18}$$

The parameter  $C$  adjusts itself to minimize the energy and setting  $dE/dC = 0$  leads to  $C = R_g / (R_c + R_g)$ . Substituting this value for  $C$  into (18), it follows that minimum value for the energy is

$$E_{\min} = \frac{Q^2}{2} \left( R_0 + \frac{R_g R_c}{R_c + R_g} \right).$$

Therefore, the total effective thermal resistance is  $R = R_0 + \mathcal{R}$  where the extra resistance  $\mathcal{R}$ , due to the combination of the gas phase and the constriction, is

$$\mathcal{R} = \frac{R_c R_g}{R_c + R_g} = \left( \frac{1}{R_c} + \frac{1}{R_g} \right)^{-1}. \tag{19}$$

The resistance  $\mathcal{R}$  is equivalent to that obtained for the simple parallel circuit comprising the resistors  $R_c$  and  $R_g$ . The re-

sistance  $\mathcal{R}$  accounts for an area  $\pi b^2$  and it gives half the total crack resistance. The conductance per area of the crack  $H_c$  is defined by

$$\frac{1}{\pi b^2 H_c} = 2\mathcal{R} = \left( \frac{1}{2R_c} + \frac{1}{2R_g} \right)^{-1}. \quad (20)$$

This expression for the crack conductance  $H_c$  includes both the effects of constriction and gas conduction. The reduced conductivity  $K_z$  can then be found using Eq. (1).

A similar resistor representation has been used previously<sup>7,8</sup> to describe the contact resistance of two nominally flat surfaces where heat conduction across the interface occurs through both asperity contacts and the gas phase between the contacts. The average distance between asperity contacts is  $2b$  and the asperity contacts are assumed to be circular with average diameter  $2a$ . The area fraction  $f$  is then defined as

$$f = \frac{\text{actual contact area}}{\text{apparent contact area}} \approx \frac{\pi a^2}{\pi b^2}.$$

This fraction, which in the present context is equivalent to the fiber volume fraction, is usually very small, typically  $f < 1\%$ . Moreover, since the solid on either side of the interface is homogeneous, the total air resistance is  $2R_g \approx (\pi b^2 h_c)^{-1}$ . The expression for  $R_g$  given in Eq. (17) is more general in the sense that it includes the case where the volume fraction is significant and the heat pipe is inhomogeneous. The constriction resistance remains to be found.

#### IV. CONSTRICTION RESISTANCE IN A HOMOGENEOUS CELL

A series form for the constriction resistance in the composite heat pipe is sought. At the outset, it is useful to have some estimate as to how many terms are necessary to give a reasonably accurate estimate. In the case where  $k_f = k_m$  and  $h = \infty$  it is possible to obtain the exact solution for the temperature in the form of a Fourier–Bessel series

$$T(r, z) = \frac{Q}{\pi b^2 k_f} \left[ z - C \sum_{i=1}^{\infty} A_i J_0(\lambda_i r) \frac{e^{-\lambda_i z}}{\lambda_i} \right], \quad (21)$$

where  $T(r, z)$  is the temperature. The terms  $J_0(\lambda_i r)$  are Bessel functions of order zero. The radial derivative of  $J_0(\lambda_i r)$  is equal to  $-\lambda_i J_1(\lambda_i r)$  so that the eigenvalues  $\lambda_i$  are found as the roots of

$$J_1(\lambda_i b) = 0. \quad (22)$$

This ensures that there is no radial flux at  $r = b$ . The boundary condition at  $z = 0$  is given in Eq. (9), and the coefficients  $A_i$  are selected to give the best fit to this boundary condition. Making use of the orthogonality properties possessed by the Bessel functions it follows that the spectral quantities  $A_i$  are

$$A_i = \left[ \frac{2J_1(\lambda_i a)}{\lambda_i a J_0^2(\lambda_i b)} \right]. \quad (23)$$

This expression for the temperature can be used in the expression for  $E_s$  to obtain

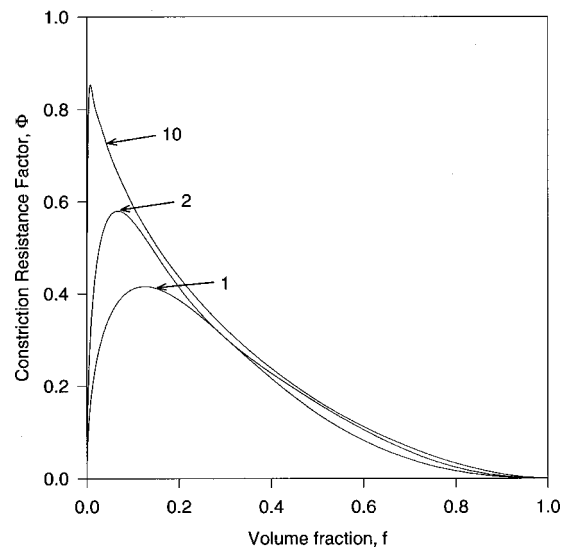


FIG. 3. Partial sums, corresponding to  $N = 1, 2,$  and  $10$  terms, for the series representing the CRF given in Eq. (27). The convergence is rapid when  $0.1 < f < 0.6$ .

$$R_c = \frac{1}{\pi b k_f} \sum_{i=1}^{\infty} \left[ \frac{2J_1(\lambda_i a)}{\lambda_i a} \frac{1}{J_0(\lambda_i b)} \right]^2 \frac{1}{\lambda_i b}. \quad (24)$$

The convergence of this series depends upon how many terms are required to approximate the “shape” of the imposed flux boundary condition. When  $a \ll b$ , many terms (hundreds), are required for the series to approximate this narrow “spike,” and, in fact, if  $a/b$  becomes sufficiently small, the series goes over to an infinite integral. This corresponds to the situation where the fibers are so far apart that they do not interact with each other. In this case, the constriction resistance tends to the limiting value<sup>9</sup>

$$R_{\infty} = \frac{8}{3\pi^2 a k_f}. \quad (25)$$

The resistance  $R_{\infty}$  is used subsequently as a normalizing factor.

To investigate the convergence of the series given in (24), it is useful to express the constriction resistance as<sup>10</sup>

$$R_c = \Phi R_{\infty}. \quad (26)$$

The term  $\Phi$  is the constriction resistance factor (CRF), and it depends on the volume fraction  $f$ . Removing the factor  $R_{\infty}$  from the series in (24) gives  $\Phi$  as

$$\Phi = \frac{3\pi\sqrt{f}}{8} \sum_{i=1}^N \left[ \frac{2J_1(\lambda_i a)}{\lambda_i a} \frac{1}{J_0(\lambda_i b)} \right]^2 \frac{1}{\lambda_i b}. \quad (27)$$

The partial sums corresponding to  $N = 1, 2,$  and  $10$  series terms are plotted as a function of  $f$  in Fig. 3. As  $f \rightarrow 0$  the quantity  $\Phi$  should approach unity. It is clear, that for values of  $f$  less than about  $0.05$ , more than  $10$  terms are required. On the other hand, for values of  $f > 0.1$ , the series has converged to virtually the exact result. The first term gives a reasonable

lower bound for the CRF when  $f > 0.2$ , and the sum of two terms gives a reasonable estimate for the CRF down to about  $f \approx 0.1$ . In almost all aligned fiber composites the volume fraction  $f > 0.1$ , and taking two terms in the series solution for the CRF is likely to give a reasonably accurate estimate.

## V. LOWER BOUND FOR THE CONSTRICTION RESISTANCE

The constriction resistance  $R_c = \Phi R_\infty$ , and this form is also applicable when the heat pipe is inhomogeneous. The CRF becomes a function of the volume fraction  $f$ , the conductivity ratio  $K$ , which is

$$K \equiv \frac{k_m}{k_f}, \quad (28)$$

and the Biot number  $Bi$ , which is defined later. To find the CRF it is necessary to obtain the thermal field within the unit cell. The inhomogeneity complicates the analysis to the point where it becomes necessary to construct some type of an approximate solution, and the calculus of variations becomes a useful tool in this regard.

To obtain the constriction resistance, the functions  $\mathcal{U}$  and  $\mathcal{V}$  need to be found. The heat flow problem that they must satisfy can be solved by maximizing the functional

$$\begin{aligned} \mathcal{J} = & -\pi \int_0^\infty \left[ k_f \int_0^a (\mathcal{U}_r^2 + \mathcal{U}_z^2) r dr + k_m \pi \int_a^b (\mathcal{V}_r^2 + \mathcal{V}_z^2) r dr \right] dz \\ & - \pi a \left[ \int_0^\infty h(z) (\mathcal{V} - \mathcal{U})^2 dz \right]_{r=a} \\ & + 2\pi \left( \int_0^a \mathcal{U} S_f r dr + \int_a^b \mathcal{V} S_m r dr \right)_{z=0}, \end{aligned} \quad (29)$$

where the subscripts  $r$  and  $z$  represent partial differentiation. The first line on the right hand side of  $\mathcal{J}$  represents the strain energy. The second term accounts for the interface condition and  $h(z)$  plays the role of a spring constant with  $(\mathcal{V} - \mathcal{U})$  being the displacement. Here, the conductance  $h(z) = h$  is taken to be constant along the length. The last term is included so that the flux boundary condition, given in Eq. (12), becomes a natural boundary condition. The functional  $\mathcal{J} = R_c Q^2 / 2$  and has units  $W^\circ C$ .

Variational calculus is based on the following idea.<sup>11</sup> It is supposed that  $\mathcal{U}(r, z)$  and  $\mathcal{V}(r, z)$  are the temperatures that cause the functional to be an extremum. These temperatures are then replaced with  $\mathcal{U}(r, z) + \xi \eta(r, z)$  and  $\mathcal{V}(r, z) + \xi \theta(r, z)$  where  $\xi$  is a small parameter. The functional then becomes a function of  $\xi$ , and attains its extremum when  $\xi = 0$ . The functions  $\eta(r, z)$ ,  $\theta(r, z)$  are arbitrary except for the restriction that they are integrable and decay to zero as  $z \rightarrow \infty$ . The differential  $d\mathcal{J}/d\xi$  can be calculated, and, after integrating this derivative by parts, it follows that when  $\xi = 0$  the derivative is

$$\begin{aligned} \frac{d\mathcal{J}}{d\xi} \Big|_0 = & 2\pi \int_0^\infty \left[ k_f \int_0^a \eta(r, z) \nabla^2 \mathcal{U} r dr + k_m \pi \int_a^b \theta(r, z) \nabla^2 \mathcal{V} r dr \right] dz - 2\pi a \int_0^\infty \{ \eta [k_f \mathcal{U}_r - h(\mathcal{V} - \mathcal{U})] \\ & - \theta [k_m \mathcal{V}_r - h(\mathcal{V} - \mathcal{U})] \}_{r=a} dz - 2\pi b \int_0^\infty \theta(b, z) [k_m \mathcal{V}_r(b, z)] dz + 2\pi \left[ \int_0^a \eta(k_f \mathcal{U}_z + S_f) r dr \right. \\ & \left. + \int_a^b \theta(k_m \mathcal{V}_z + S_m) r dr \right]_{z=0} = 0. \end{aligned} \quad (30)$$

This derivative must vanish when  $\mathcal{J}$  attains its extremum. Considering the right-hand side, the first line gives the the Euler conditions,  $\nabla^2 \mathcal{U} = \nabla^2 \mathcal{V} = 0$ , and these are the heat flow equations given in (2a) and (2b). The second, third, and fourth lines give the natural boundary conditions that must be satisfied. The quantities in the  $[\dots]$  must vanish, and are recognized as the boundary conditions given in (4a), (4b), (3), and (9), respectively. The second derivative  $d^2 \mathcal{J} / d\xi^2$  is always negative, and this indicates that the extremum corresponds to a maximum.

The variational method, as it is presented here, underestimates the constriction resistance, and this can be explained as follows. If the exact solution for the temperature field is known, then all the earlier requirements in (30) are fulfilled, and the absolute maximum value of the functional, say  $\mathcal{J}^*$ , can then be found. This quantity can be written as

$$\mathcal{J}^* = \frac{Q^2 R_c^*}{2}, \quad (31)$$

where  $R_c^*$  is the exact value for the thermal constriction resistance. In actual fact, the exact solution is not found, and some reasonable approximate form for the temperatures within fiber and matrix must be made. In the present treatment, the approximate solution does not satisfy the Euler equations and,  $\mathcal{J}$  attains a value somewhat less than the absolute maximum. The value of the functional evaluated using these approximate forms is written as

$$\mathcal{J} = \frac{Q^2 R_c}{2}. \quad (32)$$

Since  $\mathcal{J} < \mathcal{J}^*$  it follows that  $R_c < R_c^*$  so that a lower bound is obtained for the constriction resistance.

**A. Construction of the trial functions**

Here, the temperatures are represented by the two series forms

$$\begin{aligned}
 \mathcal{U}(r,z) &= \frac{Q}{\pi b^2 k_z} \sum_{i=1}^{\infty} F_i(r) Z_i(z), \\
 \mathcal{V}(r,z) &= \frac{Q}{\pi b^2 k_z} \sum_{i=1}^{\infty} G_i(r) Z_i(z).
 \end{aligned}
 \tag{33}$$

“Guesses” for the functions  $F_i(r)$ ,  $G_i(r)$  are made, and these functions are referred to as trial functions. In a homogeneous solid, the trial functions are given across the entire heat pipe by the sequence of Bessel functions  $J_0(\lambda_i r)$  where  $\lambda_i$  is a root of  $J_1(\lambda_i b) = 0$ . In comparison, within the composite, there is a discontinuity at the interface. It is necessary to construct the solution from the piecewise continuous functions  $F_i(r)$ ,  $G_i(r)$ . These functions are “patched” together at the interface so that they satisfy Eqs. (4a) and (4b). Once a suitable set of trial functions is constructed, the series (33) is used in (29), and the integration over the radial coordinate is performed. This gives the functional in terms of  $Z_i(z)$  and its derivative  $\dot{Z}_i(z)$  where the superior dot indicates differentiation. Essentially from that stage, Kantorovich’s method is used to find the functions  $Z_i(i)$  that render the functional a maximum.

According to the requirement given previously in (13), the trial functions are formulated so that they contribute no net axial flow. The requirement that the  $i$ th series term in (33) causes no net flow is given by the condition

$$k_f \int_0^a F_i(r) r dr + k_m \int_a^b G_i(r) r dr = 0.
 \tag{34}$$

The trial functions are given the forms

$$\begin{aligned}
 F_i(r) &= \sqrt{K} [\rho_i (1-f) + J_0(\lambda_i r)], \\
 G_i(r) &= \frac{1}{\sqrt{K}} [-\rho_i f + J_0(\lambda_i r)],
 \end{aligned}
 \tag{35}$$

where  $\lambda_i$  are the roots of (22). It is readily verified that these functions lead to no net axial flow along the heat pipe, i.e., they satisfy the condition (34). Moreover, the interface conditions (4a) and (4b) are satisfied if  $\rho_i$  is

$$\rho_i = \frac{k_f - k_m}{k_z} J_0(\lambda_i a) + \frac{\lambda_i a J_1(\lambda_i a)}{\text{Bi}},
 \tag{36}$$

where Bi is a Biot number which is defined by

$$\text{Bi} \equiv \frac{ah k_z}{k_f k_m}.
 \tag{37}$$

In the case where  $h$  becomes small, the quantity  $\rho_i$  becomes large in comparison with  $J_0(\lambda_i r)$  so that the trial functions lose their radial dependence. If the temperature variation in both phases is to become negligible then  $\text{Bi}_f + \text{Bi}_m \ll 1$  and this condition is commensurate with  $\text{Bi} \ll 1$ . It is again emphasized that these trial functions are only guesses at the

actual behavior. The particular form has been chosen so that they yield the exact solution when the heat pipe is homogeneous.

These trial functions are used in the functional given in (29), and the integration is performed in the radial direction. The coefficients, of the terms  $Z_i Z_j$  and  $\dot{Z}_i \dot{Z}_j$ , are equal to  $\alpha_{ij}$  and  $\beta_{ij}$ , respectively. These coefficients are

$$\frac{k_z b^2 \beta_{ij}}{2} = k_f \int_0^a F_i(r) F_j(r) r dr + k_m \int_a^b G_i(r) G_j(r) r dr,
 \tag{38a}$$

$$\begin{aligned}
 \frac{k_z \alpha_{ij}}{2} &= k_f \int_0^a \dot{F}_i(r) \dot{F}_j(r) r dr + k_m \int_a^b \dot{G}_i(r) \dot{G}_j(r) r dr \\
 &+ ah \Delta_i \Delta_j,
 \end{aligned}
 \tag{38b}$$

where  $\Delta_i \equiv G_i(a) - F_i(a)$  and  $\Delta_j \equiv G_j(a) - F_j(a)$ . After substituting the test functions in the functional and then integrating in the radial direction, it follows that, when  $i = j$ , the coefficients  $\beta_{ii}$  and  $\alpha_{ii}$  are

$$\begin{aligned}
 \beta_{ii} &= \rho_i^2 f (1-f) + 2 \rho_i f \left[ \frac{2 J_1(\lambda_i a)}{\lambda_i a} \right] \\
 &+ \frac{k_f}{k_z} [J_0(\lambda_i b)]^2 + \frac{k_m - k_f}{k_z} f \{ [J_0(\lambda_i a)]^2 + [J_1(\lambda_i a)]^2 \},
 \end{aligned}
 \tag{39a}$$

$$\begin{aligned}
 \alpha_{ii} &= \lambda_i^2 b^2 \left\{ \frac{k_f [J_0(\lambda_i b)]^2}{k_z} + \frac{2f [J_1(\lambda_i a)]^2}{\text{Bi}} \right\} + \frac{\lambda_i^2 a^2 (k_m - k_f)}{k_z} \\
 &\times \left\{ [J_0(\lambda_i a)]^2 - \frac{2 J_0(\lambda_i a) J_1(\lambda_i a)}{\lambda_i a} + [J_1(\lambda_i a)]^2 \right\}.
 \end{aligned}
 \tag{39b}$$

In the case when  $i \neq j$  the coefficients are

$$\begin{aligned}
 \beta_{ij} &= \rho_i \rho_j f (1-f) + f \left[ \frac{2 \rho_j J_1(\lambda_i a)}{\lambda_i a} + \frac{2 \rho_i J_1(\lambda_j a)}{\lambda_j a} \right] + \frac{k_m - k_f}{k_z} \\
 &\times \left[ \frac{2 a \lambda_i J_1(\lambda_i a) J_0(\lambda_j a) - 2 a \lambda_j J_1(\lambda_j a) J_0(\lambda_i a)}{b^2 (\lambda_i^2 - \lambda_j^2)} \right],
 \end{aligned}
 \tag{40a}$$

$$\begin{aligned}
 \alpha_{ij} &= \frac{2 a^2 \lambda_i \lambda_j J_1(\lambda_i a) J_1(\lambda_j a)}{\text{Bi}} + \frac{k_m - k_f}{k_z} \frac{2 \lambda_i \lambda_j}{\lambda_j^2 - \lambda_i^2} \\
 &\times [a \lambda_i J_0(\lambda_i a) J_1(\lambda_j a) - a \lambda_j J_0(\lambda_j a) J_1(\lambda_i a)].
 \end{aligned}
 \tag{40b}$$

With these definitions the functional given in (29) can be expressed, in terms of  $Z_i(z)$  and its derivative  $\dot{Z}_i(z)$ , by the forms

$$\begin{aligned}
 \mathcal{J} &= - \frac{Q^2}{\pi b^2 k_z} \left[ \frac{1}{2} \int_0^{\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{\alpha_{ij} Z_i Z_j}{b^2} + \beta_{ij} \dot{Z}_i \dot{Z}_j \right) dz \right. \\
 &\left. - \sqrt{K} \sum_{i=1}^{\infty} Q_i Z_i(0) \right].
 \end{aligned}
 \tag{41}$$

The quantities  $Q_i$  are found by considering the “work done” at  $z=0$  and are equal to

$$Q_i = \frac{2\pi}{Q\sqrt{K}} \left[ S_f \int_0^a F_i(r) r dr + S_m \int_a^b G_i(r) r dr \right]$$

$$= \rho_i(1-f) + \frac{2J_1(\lambda_i a)}{\lambda_i a}. \tag{42}$$

The flux components  $S_f$  and  $S_m$  are given in Eq. (10).

At this juncture, it is prudent to consider the available options to maximize the functional in (41). The Ritz method, which is the simplest technique, does not give the best results. In the Ritz method, a guess is made for the functions, and for the sake of discussion say that  $Z_i(z)$  is

$$Z_i(z) = \frac{C_i e^{-\lambda_i z}}{\lambda_i}, \tag{43}$$

where  $\lambda_i$  are the roots of (22). Using these functions, the integral in (41) can be evaluated, and  $\mathcal{J}$  is found as a function of the coefficients, i.e.,  $\mathcal{J}(C_1, C_2, \dots)$ . Setting  $\partial\mathcal{J}/\partial C_i = 0$ , it is possible to find the coefficients  $C_i$  which maximize the functional. In general the heat pipe is inhomogeneous, and the function given in (43) may greatly differ from the actual (exact) behavior. The second option, which uses the method of Kantorovich, is semi-analytical in the sense that the exact functions,  $Z_1(z)$ ,  $Z_2(z)$  etc., that maximize  $\mathcal{J}$  are found. However, the solution is not completely analytical, because the trial functions,  $F_i(r)$  and  $G_i(r)$ , are not exact. Kantorovich’s method gives more accurate results than the Ritz’ method, and the expressions for  $Z_i(z)$  are valid for all values of the thermal parameters  $K$  and  $Bi$ . In the subsequent analysis, the first series term is presented. The two term solution is somewhat longer, and is given in the Appendix.

**B. Lower bound for the constriction resistance factor**

Using only the first series term, the functional in Eq. (41) reduces to the form

$$\mathcal{J}_1 = - \frac{Q^2}{\pi b^2 k_z} \left[ \frac{1}{2} \int_0^\infty \left( \frac{\alpha_{11} Z_1^2}{b^2} + \beta_{11} \dot{Z}_1^2 \right) dz - \sqrt{K} Q_1 Z_1(0) \right].$$

Finding the function  $Z_1(z)$  that maximizes the functional is a standard problem in variational calculus.<sup>11</sup> This function must satisfy the Euler condition

$$\frac{d^2 Z_1}{dz^2} - \gamma_1^2 Z_1 = 0, \tag{44}$$

where the eigenvalue  $\gamma_1$  is

$$\gamma_1 = \frac{1}{b} \sqrt{\frac{\alpha_{11}}{\beta_{11}}}. \tag{45}$$

The natural boundary condition, which is also necessary to maximize the functional, is  $\beta_{11} \dot{Z}_1(0) = -\sqrt{K} Q_1$ . The solution to (44), satisfying this natural boundary condition, gives the function

$$Z_1(z) = \frac{Q_1 \sqrt{K}}{\beta_{11}} \frac{\exp\{-\gamma_1 z\}}{\gamma_1}$$

that causes  $\mathcal{J}_1$  to attain its maximum value. Finally, the constriction resistance is

$$R_c = \frac{\bar{U}}{Q} = \frac{\sqrt{K} Z_1(0) Q_1}{\pi b^2 k_z}.$$

Upon removing the factor  $R_\infty$ , the lower bound for the CRF is

$$\Phi_1 = \frac{3\pi k_m \sqrt{f}}{8k_z} \left( \frac{Q_1}{\sqrt{\beta_{11}}} \right)^2 \frac{1}{\gamma_1 b}, \tag{46}$$

where the subscript ‘‘1’’ indicates the first term. If the heat pipe is homogeneous this expression for  $\Phi_1$  reduces to the first series term of the exact solution given in Eq. (27). The solution to find  $\Phi_2$ , which proceeds along the same lines, is given in the Appendix.

**VI. UPPER BOUND FOR THE CONSTRICTION RESISTANCE**

In deriving an upper bound for the constriction resistance it is useful to consider the analogous problem in elasticity where bounds for the compliance are found. To obtain a lower bound on the compliance, a strain field is specified. This strain is uniform in Voigt’s model. The elastic energy is then found using this strain field and an effective stiffness is obtained. In Voigt’s model, the stiffest phase dominates the calculation, and in the case where there is a rigid phase the method fails. In contrast, an upper bound for the compliance can be found by specifying a stress field. This stress is uniform in Reuss’ model. The strain energy is found using this stress. Reuss’ model is dominated by the most compliant phase, and in the case where there is a cavity, this method fails. In the thermal problem, an upper bound for the constriction resistance is obtained by specifying an appropriate flux field. This upper bound is not very useful when one of the phases has an inordinately high thermal resistance. Specifically, the case where the Biot number  $Bi \ll 1$  is troublesome.

To begin, let the flux components in the fiber be defined as  $\Psi_r$ ,  $\Psi_z$ , and those in the matrix by  $\Omega_r$ ,  $\Omega_z$ , so that

$$\begin{pmatrix} \Psi_r & \Psi_z \\ \Omega_r & \Omega_z \end{pmatrix} = \begin{pmatrix} k_f \mathcal{U}_r & k_f \mathcal{U}_z \\ k_m \mathcal{V}_r & k_m \mathcal{V}_z \end{pmatrix}.$$

The omission of the negative sign does not affect the subsequent analysis. The flux components are considered as new variables and, following Arthurs,<sup>12</sup> the Hamiltonians  $\mathcal{F}$ ,  $\mathcal{M}$  are defined as

$$\mathcal{F} = \frac{\Psi_r^2 + \Psi_z^2}{2k_f} \quad \text{and} \quad \mathcal{M} = \frac{\Omega_r^2 + \Omega_z^2}{2k_m}.$$

The functional in Eq. (29) can be written in its canonical form

$$\begin{aligned} \mathcal{J} = & -2\pi \int_0^\infty \left[ \int_0^a (\Psi_r \mathcal{U}_r + \Psi_z \mathcal{U}_z - \mathcal{F}) r dr \right. \\ & \left. + \int_a^b (\Omega_r \mathcal{V}_r + \Omega_z \mathcal{V}_z - \mathcal{M}) r dr \right] dz - \pi h a \\ & \times \int_0^\infty [\mathcal{V}(a, z) - \mathcal{U}(a, z)]^2 dz + 2\pi \\ & \times \left[ \int_0^a \mathcal{U}(r, 0) S_f r dr + \int_a^b \mathcal{V}(r, 0) S_m r dr \right]. \end{aligned} \tag{47}$$

At this stage, the original functional can be recovered if the flux components are replaced with the temperature derivatives. On the other hand, if the temperatures  $\mathcal{U}$  and  $\mathcal{V}$  can be eliminated, the complementary functional, say  $\mathcal{I}$ , is obtained. To obtain this complementary functional, the volume integrals in (47) are integrated by parts to obtain

$$\begin{aligned} \mathcal{I} = & 2\pi \int_0^\infty \left( \int_0^a \{ \mathcal{U}[(r\Psi_r)_r + (r\Psi_z)_z] + r\mathcal{F} \} dr \right. \\ & \left. + \int_a^b \{ \mathcal{V}[(r\Omega_r)_r + (r\Omega_z)_z] + r\mathcal{M} \} dr \right) dz \\ & + \pi a \left\{ \int_0^\infty [2\mathcal{V}\Omega_r - 2\mathcal{U}\Psi_r - h(\mathcal{V} - \mathcal{U})^2] dz \right\}_{r=a} \\ & - 2\pi b \left( \int_0^\infty \mathcal{V}\Omega_r dz \right)_{r=b} + 2\pi \left[ \int_0^a \mathcal{U}(\Psi_z + S_f) r dr \right. \\ & \left. + \int_a^b \mathcal{V}(\Omega_z + S_m) r dr \right]_{z=0}. \end{aligned} \tag{48}$$

In the first line, the coefficients of  $\mathcal{U}$  and  $\mathcal{V}$  contained in the  $[\dots]$ , vanish if the divergence of the flux vanishes; this is analogous to stress equilibrium. The first term in the second line gives the boundary condition along  $r=a$  and equilibrium requires that  $\Psi_r = \Omega_r$ . The interface boundary condition,  $h(\mathcal{V} - \mathcal{U}) = \Psi_r$ , can then be used to replace the temperatures with  $\Psi_r$ . The second term requires that the adiabatic condition at  $r=b$  must be satisfied if  $\mathcal{V}$  is to be eliminated. The last line gives the boundary conditions at  $z=0$ , and the quantities in the  $[\dots]$  vanish if the axial flux components satisfy the applied boundary conditions. It should be noted that these requirements on the flux field are the natural conditions which are given in Eq. (30). The complementary functional  $\mathcal{I}$  is then found

$$\begin{aligned} \mathcal{I} = & 2\pi \int_0^\infty \left[ \int_0^a \frac{\Psi_r^2 + \Psi_z^2}{2k_f} r dr + \frac{a\Psi_r^2(a, z)}{2h} \right. \\ & \left. + \int_a^b \frac{\Omega_r^2 + \Omega_z^2}{2k_m} r dr \right] dz = \frac{Q^2 R_c}{2}. \end{aligned} \tag{49}$$

It is clear that  $\mathcal{I}$  is always positive and assumes its minimum value when the flux components represent the exact solution. The exact minimum is equal to the quantity  $\mathcal{J}^* = Q^2 R_c^*/2$  which is given in (31). Since  $\mathcal{I} > \mathcal{J}^*$ , it follows that  $R_c > R_c^*$ , and this functional can be used to obtain an upper bound for the constriction resistance.

A flow field satisfying the requirements of Eq. (48) can be found by the use of a suitably chosen Harmonic potential function, say  $W(r, z)$ , which is taken in the form of a Fourier-Bessel series

$$W(r, z) = \sum_{n=1}^\infty B_n J_0(\lambda_n r) \frac{e^{-\lambda_i z}}{\lambda_i}. \tag{50}$$

The insulated condition at  $r=b$  is taken into account by finding the roots  $\lambda_n$  so that  $J_1(\lambda_n b) = 0$ . The flux components are taken as

$$\begin{pmatrix} \Psi_r & \Psi_z \\ \Omega_r & \Omega_z \end{pmatrix} = \begin{pmatrix} W_r & W_z \\ W_r & W_z \end{pmatrix}. \tag{51}$$

If the coefficients  $B_n$  are to satisfy the flux boundary condition at  $z=L$ , as given in (12), then they must satisfy

$$\frac{\partial W}{\partial z} \Big|_L = - \sum_{n=1}^\infty B_n J_0(\lambda_n r) = \begin{cases} -S_f & (0 < r < a) \\ -S_m & (a < r < b) \end{cases}. \tag{52}$$

Each side of this equation is multiplied by  $rJ_0(\lambda_p r)$ . and then integrated over  $0 < r < b$ , to give the Fourier coefficients  $B_n$  as

$$B_n = P_m \left[ \frac{2J_1(\lambda_n a)}{\lambda_n a J_0^2(\lambda_n b)} \right]. \tag{53}$$

Using these definitions in Eq. (49), the constriction resistance  $R_c$  is

$$\begin{aligned} R_c = & \frac{2\pi}{Q^2} \int_0^\infty \left[ \int_0^a \frac{W_r^2 + W_z^2}{k_f} r dr + \int_a^b \frac{W_r^2 + W_z^2}{k_m} r dr \right. \\ & \left. + \frac{aW_r^2(a, z)}{h} \right] dz. \end{aligned} \tag{54a}$$

Because  $\nabla^2 W = 0$  it is possible to convert this volume integral into a surface integral by the use of integration by parts. The result of this process is

$$\begin{aligned} R_c = & \frac{2\pi}{Q^2} \left[ S_f \int_0^a \frac{W(r, 0)}{k_f} r dr + S_m \int_a^b \frac{W(r, 0)}{k_m} r dr \right] \\ & + \frac{2\pi}{Q^2} \int_0^\infty \left[ \frac{W(a, z)}{k_f} - \frac{W(a, z)}{k_m} + \frac{W_r(a, z)}{h} \right] \\ & \times a W_r(a, z) dz. \end{aligned} \tag{54b}$$

The heat flow is continuous and is given by the partial derivatives of  $W$ . In comparison, the temperatures  $\mathcal{U} = W/k_f$ ,  $\mathcal{V} = W/k_m$  are discontinuous across  $r=a$ . According to the interface condition, the temperatures should satisfy the condition  $h(\mathcal{V} - \mathcal{U}) = k_f \mathcal{U}_r$ , and the second line in (54b) should (if the solution is exact), vanish. With the choice given for  $W$  in (50) this ‘‘compatibility’’ condition is not in general fulfilled so that  $R_c > R_c^*$ . The exceptional case arises if  $k_f = k_m$  and  $h = \infty$  and the expression for  $R_c$  is then exact. After some algebra, the expression in (54b) can be evaluated. Upon factoring out  $R_\infty$  the CRF is

$$\Phi_u = \frac{k_m}{k_z} \left[ \Phi + \frac{3\pi f}{4} \sum_{n=1}^\infty \sum_{i=1}^\infty \frac{A_n \rho_n A_i J_1(\lambda_i a)}{b^2 \lambda_n (\lambda_n + \lambda_i)} \right], \tag{54c}$$

where  $\Phi$ ,  $A_n$ , and  $\rho_n$  are given in (27), (23), and (36) respectively. The subscript ‘‘u’’ indicates the upper bound, and the accuracy of this bound is governed by how well the flux field is approximated by the gradient of  $W(r,z)$ . In some situations, for example when  $h \rightarrow 0$ , the flux distribution given by  $\mathbf{q} = \nabla W$  is not very realistic and the bound is not useful. No attempt has been made to correct the upper bound of this defect.

**VII. LIMITING CASES FOR EXTREME THERMAL PROPERTIES**

Before a discussion of the results, it is convenient to present the limiting behavior of the CFR corresponding to cases where the thermophysical properties attain extreme values.

**A. Perfect thermal contact along the interface**

When the interface is perfect, there are two limiting cases corresponding to either very large, or very small, values of the conductivity ratio  $K$ . First, when  $K \rightarrow \infty$ , the radial variation in the matrix temperature becomes negligible, and it is reasonable to set  $\mathcal{U}(a,z) = 0$  as a condition at  $r = a$ . Moreover, as  $K \rightarrow \infty$ , the flux boundary condition, Eq. (12), specifies that  $k_f \mathcal{U}_z = -S_f \rightarrow -Q/\pi a^2$ . The function  $\mathcal{U}(r,z)$  under these conditions is then

$$\mathcal{U}(r,z) = \frac{2Q}{\pi a^2 k_f} \sum_{n=1}^{\infty} \frac{J_0(\Lambda_n r)}{\Lambda_n a J_1(\Lambda_n a)} \frac{\exp(-\Lambda_n z)}{\Lambda_n}, \quad (55)$$

where the eigenvalues  $\Lambda_n$  are the roots of  $J_0(\Lambda_n a) = 0$ . At  $z = 0$  the average temperature rise of the fiber is equal to  $\bar{\mathcal{U}}$  and the constriction resistance is  $R_c = \bar{\mathcal{U}}/Q$ . After doing the integration over  $0 < r < a$  to find the average value, and factoring out  $R_\infty$ , the limiting value for the CRF is

$$\begin{aligned} \Phi_K &\equiv \lim_{K \rightarrow \infty} \Phi = \frac{3\pi}{2} \sum_{n=1}^{\infty} \left( \frac{1}{\Lambda_n a} \right)^3 \\ &= \frac{3\pi}{2} \left( \frac{1}{2.4048^3} + \frac{1}{5.5201^3} + \dots \right) \approx 0.38. \end{aligned} \quad (56)$$

The second extreme situation occurs when  $K \rightarrow 0$ . The limiting value for the CRF can be found by noting that, far from the crack where the flow is purely axial, the ratio of the matrix flux  $Q_m$  to the total flux  $Q$  is

$$\frac{Q_m}{Q} = \frac{\pi(b^2 - a^2)P_m}{P\pi b^2} = \frac{K(1-f)}{K(1-f) + f}.$$

For any fixed value of  $f$ , the conductivity ratio  $K$  can be taken sufficiently small, so that  $Q_m/Q \approx 0$ . The limiting value of the CRF is then

$$\Phi_0 \rightarrow 0. \quad (57)$$

All the heat is then carried by the fiber and no readjustment of the heat flow is required to cross the crack.

**B. Asymptotic behavior of a weakly coupled interface**

In the extreme case wherein the Biot number becomes sufficiently small, the radial temperature variation in both

phases becomes negligible, and the analysis can be reduced to a one-dimensional problem. Assuming that  $\mathcal{U}$  and  $\mathcal{V}$  are functions only of  $z$ , an elementary heat flow balance leads to the pair of equations

$$\begin{aligned} \pi a^2 \left( k_f \frac{d^2 \mathcal{U}}{dz^2} \right) - 2\pi a h (\mathcal{U} - \mathcal{V}) &= 0, \\ \pi (b^2 - a^2) \left( k_m \frac{d^2 \mathcal{V}}{dz^2} \right) + 2\pi a h (\mathcal{U} - \mathcal{V}) &= 0. \end{aligned}$$

It is readily verified that the solution to these equations, which satisfies the boundary condition (12), is of the form

$$\begin{aligned} \mathcal{U}(z) &= \frac{S_f \exp(-\gamma_w z)}{k_f \gamma_w}, \\ \mathcal{V}(z) &= \frac{S_m \exp(-\gamma_w z)}{k_m \gamma_w}, \end{aligned}$$

where the decay rate  $\gamma_w$  is

$$\gamma_w = \frac{1}{b} \sqrt{\frac{2\text{Bi}}{f(1-f)}}.$$

The subscript ‘‘w’’ indicates ‘‘weak’’ coupling across the fiber/matrix interface. The constriction resistance

$$R_c = \frac{\mathcal{U}(0)}{Q} = \frac{1}{\pi b k_z} \frac{(1-f)k_m}{f k_f} \sqrt{\frac{f(1-f)}{2\text{Bi}}}.$$

After factoring out  $R_\infty$ , the CRF is found as

$$\Phi_w = \frac{3\pi k_m (1-f)}{8k_z} \sqrt{\frac{1-f}{2\text{Bi}}}. \quad (58)$$

This expression for the CRF represents the asymptotic behavior when the interface is weakly conducting. According to elementary books on heat transfer,<sup>13</sup> the internal temperature variation can be neglected if the Biot number  $\text{Bi} < 0.1$ . This general prediction seems to be approximately true as subsequently shown.

**VIII. NUMERICAL RESULTS AND DISCUSSION**

Here some numerical results, obtained from the variational method, are compared with those obtained by L.H. The L.H. model yields the formula

$$\Phi_{\text{LH}} = 3\pi \sqrt{\left[ \frac{k_m(1-f)}{8k_z} \right]^3 \left[ \frac{4 + \text{Bi}_f}{\text{Bi}_f} \right]}. \quad (59)$$

The subscript ‘‘LH’’ denotes their model. In formulating their analysis, the radial variation in the matrix temperature was considered to be negligible. For this to be true, the thermal resistance of the matrix  $R_m \approx (1-f)/k_m$ , must be somewhat less than the sum of the fiber resistance  $R_f \approx f/k_f$ , plus the interface resistance  $R_i = 1/ha$ . Setting  $R_f + R_i \geq R_m$ , yields the result

$$\frac{Kf}{1-f} + \frac{K}{\text{Bi}} + \frac{f}{\text{Bi}(1-f)} \geq 1. \quad (60)$$

As illustrated later, this inequality seems to give a reasonable estimate for the region of validity of the L.H. model.

**A. Perfect thermal contact along the interface**

To begin the discussion, a thermally perfect interface is considered. The Biot number,  $Bi \rightarrow \infty$ , and the expression for  $\Phi_{LH}$  reduces to the simple form

$$\Phi_{LH} = 3\pi \left[ \frac{k_m(1-f)}{8k_z} \right]^{3/2} \tag{61}$$

Figures 4(a), 4(b), and 4(c) show the behavior of all the various estimates ( $\Phi_{LH}$ ,  $\Phi_1$ ,  $\Phi_2$ , and  $\Phi_u$ ), as a function of the volume fraction  $f$ . In practice, the volume fraction rarely exceeds  $f=0.6$  so the abscissa is truncated at that value. Each figure corresponds to a given value of  $K$ , and values of  $K=1, 10$ , and  $0.1$  are used. The dashed line shows  $\Phi_{LH}$ , the thin solid line shows  $\Phi_1$ , and the two heavy lines show  $\Phi_2, \Phi_u$ . The exact solution  $\Phi^*$ , which is in the range  $\Phi_2 \leq \Phi^* \leq \Phi_u$ , lies between the two heavy lines. For values of volume fraction in the range  $0.1 < f < 0.6$ , the bounds are fairly close together so that  $\Phi^*$  can be estimated to within a relatively small error.

Figure 4(a) shows the behavior of the CRF when the conductivity ratio  $K=1$ . The upper bound is identical to the line  $N=10$  in Fig. 3. The lower bounds correspond to the partial sums,  $N=1$  and  $2$  in Fig. 3. For values of  $f > 0.5$  the L.H. model is in reasonable agreement with the bounds and this is consistent with the inequality (60).

Figure 4(b) shows the behavior of the CRF corresponding to  $K=10$ . Since  $K$  is fairly large, the radial variation in the matrix temperature becomes relatively “flat” in comparison with that of the fiber. According to (60), the estimate for  $\Phi_{LH}$  is valid when  $f \geq 1/9$ , and it is clear from the figure that the expression for  $\Phi_{LH}$  is in reasonable agreement with the other estimates. The horizontal double-dashed line, shows the limiting value  $\Phi_K$  which is described by Eq. (56).

Figure 4(c) shows the behavior of the CRF corresponding to  $K=0.1$ . Since  $K$  is small, the radial variation within the matrix becomes large in comparison with that of the fiber. The L.H. model does not give a good estimate of the CRF for the range of  $f$  shown in the figure. Moreover, the inequality (60) suggests that  $f > 10/11$  is required for the L.H. model to give a reasonable estimate.

**B. Imperfect thermal contact along the interface**

Figures 5(a), 5(b), and 5(c) show the effect of the Biot number upon the CRF. The volume fraction  $f=0.3$ , and values of  $K=1, 10$  and  $0.1$  are used as labeled in the figures. The three solid lines represent the bounds, and when  $f=0.3$  the two lower bounds almost coincide. As previously discussed, the upper bound becomes inaccurate when  $Bi$  becomes small, and this appears to happen when  $Bi < 1$ . The dashed line shows the L.H. model and, according to the inequality (60),  $\Phi_{LH}$  is expected to be useful when the Biot number is sufficiently small so that

$$Bi \leq \frac{3+7K}{7-3K}$$

For  $K > 7/3$ , the estimate  $\Phi_{LH}$  is expected to be reasonably accurate for all values of  $Bi$ . The asymptotic behavior  $\Phi_w$  is shown by the straight double-dashed line, and, as expected,

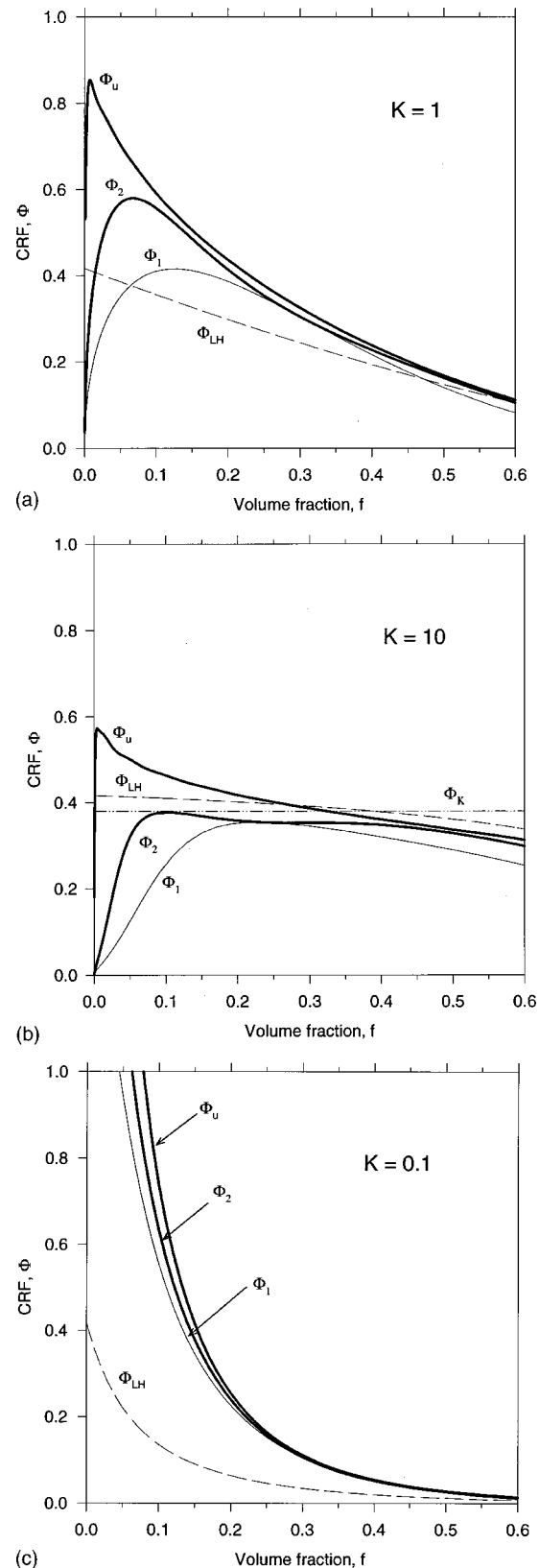


FIG. 4. Comparisons of the various bounds on the CRF, along with the L.H. model, when  $Bi = \infty$ . The following values of the conductivity ratio are used: (a)  $K=1$ , (b)  $K=10$ , and (c)  $K=0.1$ .

the CRF tends to this asymptotic limit when  $Bi < 0.1$  or so. In Fig. 5(a) the conductivity ratio  $K=1$ , and the inequality predicts that  $\Phi_{LH}$  is useful when  $Bi < 2.5$ . In Fig. 5(b) the value of  $K=10$  and the CRF given by  $\Phi_{LH}$  can barely be distin-

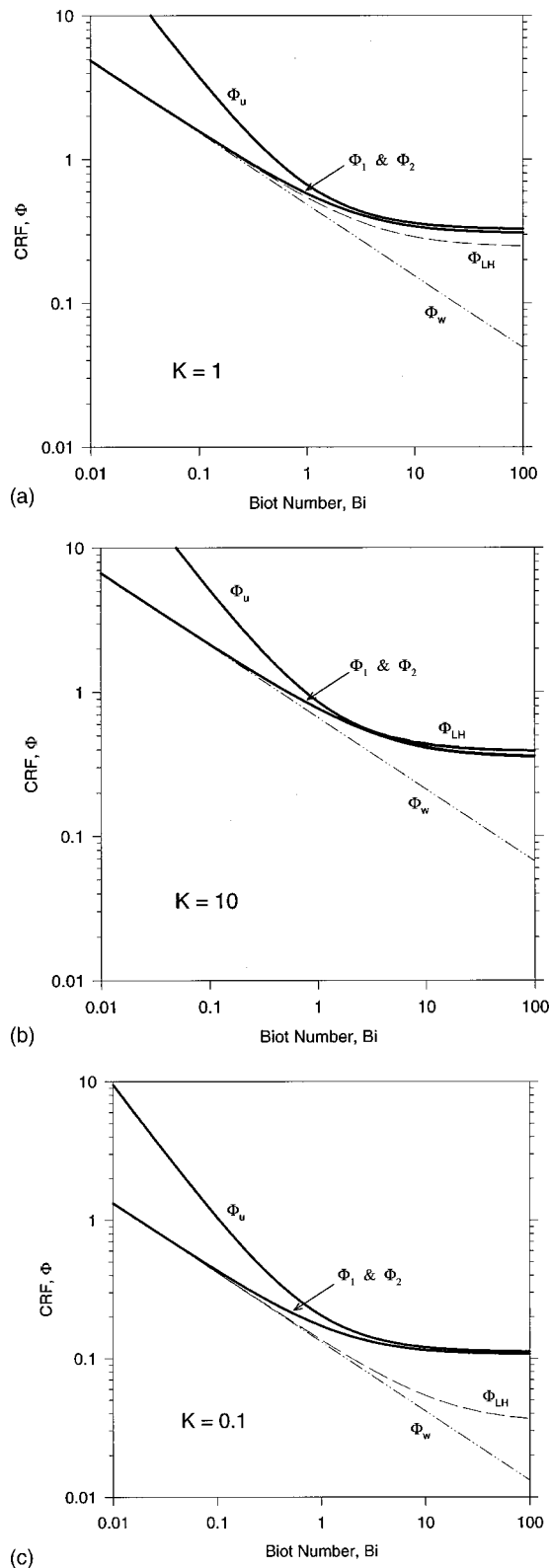


FIG. 5. Effect of the Biot number upon the CRF for composites with the volume fraction  $f=0.3$ . The following values of the conductivity ratio are used: (a)  $K=1$ , (b)  $K=10$ , and (c)  $K=0.1$ .

guished from the solid lines. In Fig. 5(c) the conductivity ratio  $K=0.1$  and the inequality predicts that the L.H. model is valid when  $Bi \leq 0.55$ . The figure indicates that this is approximately true.

### IX. CONCLUDING REMARKS

It has been demonstrated that the thermal resistance of a bridged crack can be obtained from a simple parallel resistor model, with one of the resistors representing the thermal resistance of the gas phase within the crack and the other resistor representing the constriction resistance. The gas resistance  $R_g$  is related to the various geometric and thermal properties in a straightforward manner as given in Eq. (17). Furthermore, the use of this parallel resistance  $\mathcal{R}$  gives results that are consistent with the analysis obtained by L.H.

Determination of the constriction resistance is considerably more challenging, and has been the main focus of this article. Upper and lower bound estimates have been obtained using an approach based on variational calculus, and emphasis has been placed on the analogy between problems involving heat flow and elasticity in multiphase systems. Upon comparing the bounds with one another, and with other limiting values of the constriction resistance, the two-term lower bound solution emerges as the most accurate over essentially the entire range of parameter values of interest (the notable exception being the case where the fiber volume fraction is very small). The one-term lower bound solution is somewhat simpler yet provides reasonable estimates over the entire range of conductivity ratios and Biot numbers when the fiber volume fraction is moderately high ( $f \geq 0.2$ ).

The solution for the constriction resistance stemming from the model of L.H. is accurate within the domain predicted by the approximate inequality (60). Outside this domain the L.H. model can be in error by a significant amount and hence the solutions presented in this article are required.

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### APPENDIX: TWO TERM LOWER BOUND SOLUTION

Using two series terms, along with the definitions given above for  $\alpha_{ij}$  and  $\beta_{ij}$ , the functional in (29), after integration over the radius, reduces to the form

$$\begin{aligned} \mathcal{J}_2 = & -\frac{Q^2}{2\pi b^2 k_z} \int_0^\infty \frac{1}{b^2} [\alpha_{11} Z_1^2(z) + 2\alpha_{12} Z_1(z) Z_2(z) \\ & + \alpha_{22} Z_2^2(z)] dz - \frac{Q^2}{2\pi b^2 k_z} \int_0^\infty [\beta_{11} \dot{Z}_1^2(z) \\ & + 2\beta_{12} \dot{Z}_1(z) \dot{Z}_2(z) + \beta_{22} \dot{Z}_2^2(z)] dz \\ & + \frac{Q^2}{\pi b^2 k_z} \{ \sqrt{K} [Q_1 Z_1(0) + Q_2 Z_2(0)] \}. \end{aligned} \quad (A1)$$

The problem of finding the two dependent variables,  $Z_1(z)$  and  $Z_2(z)$ , which maximize  $\mathcal{J}_2$  is a standard problem in

variational calculus. In the range  $0 < z < \infty$ , these functions must satisfy the Euler equations. In this case there are two coupled differential equations which are

$$\begin{aligned} b^2 \beta_{11} \ddot{Z}_1 + b^2 \beta_{12} \ddot{Z}_2 &= \alpha_{11} Z_1 + \alpha_{12} Z_2, \\ b^2 \beta_{12} \ddot{Z}_1 + b^2 \beta_{22} \ddot{Z}_2 &= \alpha_{12} Z_1 + \alpha_{22} Z_2. \end{aligned} \tag{A2}$$

Similarly, there are two natural boundary conditions at  $z = 0$  which must be satisfied to maximize the functional. These boundary conditions are

$$\begin{aligned} \beta_{11} \dot{Z}_1(0) + \beta_{12} \dot{Z}_2(0) &= -Q_1 \sqrt{K}, \\ \beta_{12} \dot{Z}_1(0) + \beta_{22} \dot{Z}_2(0) &= -Q_2 \sqrt{K}. \end{aligned} \tag{A3}$$

Finally, the functions  $Z_1(z)$  and  $Z_2(z)$  must be constructed so that they decay as  $z \rightarrow \infty$ .

To obtain the solution, it is noted that the differential equations in (A2) are linear with constant coefficients. It then follows that forms, of the type  $Z_1(z) = De^{\gamma z}$  and  $Z_2(z) = Ee^{\gamma z}$ , can represent a solution if the coefficients and the roots are appropriately chosen. Substituting these forms in Eq. (A2) yields the matrix equation

$$\begin{pmatrix} b^2 \gamma^2 \beta_{11} - \alpha_{11} & b^2 \gamma^2 \beta_{12} - \alpha_{12} \\ b^2 \gamma^2 \beta_{12} - \alpha_{12} & b^2 \gamma^2 \beta_{22} - \alpha_{22} \end{pmatrix} \begin{pmatrix} D \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{A4}$$

For nonzero values of  $D$  and  $E$  the roots  $\gamma$  must be found so that the determinant of the matrix vanishes. The two roots for  $\gamma^2$  are found from the quadratic

$$b^4 \gamma^4 - b^2 \gamma^2 B + C = 0, \tag{A5}$$

where  $B$  and  $C$  are given by

$$\begin{aligned} B &= \frac{\beta_{11} \alpha_{22} + \beta_{22} \alpha_{11} - 2 \beta_{12} \alpha_{12}}{\beta_{11} \beta_{22} - \beta_{12}^2}, \\ C &= \frac{\alpha_{11} \alpha_{22} - \alpha_{12}^2}{\beta_{11} \beta_{22} - \beta_{12}^2}. \end{aligned}$$

There are four roots that occur as  $\pm \gamma_1$  and  $\pm \gamma_2$ . These roots are found from Eq. (A5):

$$\gamma_{1,2}^2 = \frac{B \pm \sqrt{B^2 - 4C}}{2b^2}.$$

The labeling of the roots is such that subscripts correspond to the first and second eigenvalues, respectively, i.e.,  $\gamma_1 < \gamma_2$ . If these roots are used in the matrix Eq. (A4) then the determinant vanishes and the two equations are dependent. This implies that the ratio of  $D$  to  $E$  is specified. Adding the two rows gives

$$\begin{aligned} D[b^2 \gamma^2 (\beta_{11} + \beta_{12}) - (\alpha_{11} + \alpha_{12})] + E[b^2 \gamma^2 (\beta_{12} + \beta_{22}) \\ - (\alpha_{12} + \alpha_{22})] = 0. \end{aligned}$$

This condition can be satisfied upon setting,  $D = \zeta F$  and  $E = \eta F$ , where

$$\zeta = [b^2 \gamma^2 (\beta_{12} + \beta_{22}) - (\alpha_{12} + \alpha_{22})],$$

$$\eta = -[b^2 \gamma^2 (\beta_{11} + \beta_{12}) - (\alpha_{11} + \alpha_{12})].$$

There are two values for  $\zeta$  and  $\eta$  corresponding to the roots of  $\gamma_1$  and  $\gamma_2$ . Retaining the appropriate roots, so that the temperature decays as  $z$  increases, the functions  $Z_1(z)$  and  $Z_2(z)$  are found as

$$\begin{aligned} Z_1(z) &= \sqrt{K} \left[ \frac{F \zeta_1 \exp(-\gamma_1 z)}{\gamma_1} + \frac{G \zeta_2 \exp(-\gamma_2 z)}{\gamma_2} \right], \\ Z_2(z) &= \sqrt{K} \left[ \frac{F \eta_1 \exp(-\gamma_1 z)}{\gamma_1} + \frac{G \eta_2 \exp(-\gamma_2 z)}{\gamma_2} \right]. \end{aligned} \tag{A6}$$

Finally, the coefficients  $F$  and  $G$  are found by using the boundary conditions given in (A3):

$$\begin{aligned} F &= \frac{Q_1 (\eta_2 \beta_{22} + \zeta_2 \beta_{12}) - Q_2 (\eta_2 \beta_{12} + \zeta_2 \beta_{11})}{(\beta_{11} \beta_{22} - \beta_{12}^2) (\eta_2 \zeta_1 - \eta_1 \zeta_2)}, \\ G &= \frac{-Q_1 (\eta_1 \beta_{22} + \zeta_1 \beta_{12}) + Q_2 (\eta_1 \beta_{12} + \zeta_1 \beta_{11})}{(\beta_{11} \beta_{22} - \beta_{12}^2) (\eta_2 \zeta_1 - \eta_1 \zeta_2)}. \end{aligned} \tag{A7}$$

Thus, both the Euler conditions and the boundary conditions are satisfied by setting  $Z_1(z)$  and  $Z_2(z)$  equal to the forms given in (A6).

It can be shown, using integration by parts on (A1), that the constriction resistance is

$$R_c = \frac{\sqrt{K}}{\pi b^2 k_z} [Z_1(0) Q_1 + Z_2(0) Q_2]. \tag{A8}$$

Upon removing the factor  $R_\infty$ , the first two terms of the constriction resistance factor are obtained

$$\Phi_2 = \frac{3 \pi k_m \sqrt{f}}{8 k_z} \left[ \frac{F (Q_1 \zeta_1 + Q_2 \eta_1)}{\gamma_1 b} + \frac{G (Q_1 \zeta_2 + Q_2 \eta_2)}{\gamma_2 b} \right]. \tag{A9}$$

When the heat pipe is homogeneous, this expression reduces to the first two terms of the exact solution given in Eq. (27).

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